Multivariate linear discrete-time deterministic models (1)

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Annual plant example

Parameters: seed production γ, overwinter survival σ, first-year germination α, second-year germination β. Homogeneous second-order linear model: $N(t) = \gamma\alpha\sigma N(t-1) + \gamma\sigma^2(1-\alpha)\beta N(t-2)$.

For homogeneous linear equations of any order, 0 is an equilibrium. Find other eq. by (1) plugging in trial solution $C\lambda^t$; (2) dividing through by $C\lambda^{n-1}$; (3) solve for λ by finding roots of *characteristic equation*; (4) possibly plugging in initial conditions ($N(0)$, $N(-1)$, ...) to solve for constants c_i in particular solution $N(t) = \sum_i c_i \lambda_i^t$ (ignoring repeated-root case).

In this case with $a = \gamma\alpha\sigma$, $b = \gamma\sigma^2(1-\alpha)\beta$, we have $\lambda^2 - a\lambda - b = 0 \rightarrow \lambda = (a \pm \sqrt{a^2+4b})/2$. Given that $\lambda > 1 \leftrightarrow a + b > 1$, population grows if $\gamma > 1/(\alpha\sigma + \beta(1-\alpha)\sigma^2)$.

Or we can set this up as a matrix equation:

$$\begin{pmatrix} P(t+1) \\ S(t+1) \end{pmatrix} = \begin{pmatrix} \gamma\alpha\sigma & \sigma\beta \\ \gamma\sigma(1-\alpha) & 0 \end{pmatrix} \begin{pmatrix} P(t) \\ S(t) \end{pmatrix}$$

Basic model: $\boldsymbol{x}(t+1) = \boldsymbol{M}\boldsymbol{x}(t)$.

Alternative case, juvenile/adult model: fractions $\{s_J, s_A\}$ of juveniles and adults survive; adults have f offspring each (on average); surviving juveniles become adults. So $A(t+1) = s_A A(t) + s_J J(t)$, $J(t+1) = fA(t)$ *or* $A(t+1) = s_A A(t) + s_J f A(t-1)$. This can be written as a matrix equation,

$$\begin{bmatrix} J(t+1) \\ A(t+1) \end{bmatrix} = \begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix} \begin{bmatrix} J(t) \\ A(t) \end{bmatrix}$$

Fixed points

$\boldsymbol{x}\star$ is a fixed point if $\boldsymbol{x}\star = \boldsymbol{M}\boldsymbol{x}\star$, $\boldsymbol{0} = (\boldsymbol{M} - \boldsymbol{I})\boldsymbol{x}\star$ where $\boldsymbol{I}$ is the identity matrix. The null space of $\boldsymbol{M} - \boldsymbol{I}$ has all the fixed points. If $\boldsymbol{M} - \boldsymbol{I}$ is invertible, we find $\boldsymbol{0} = (\boldsymbol{M} - \boldsymbol{I})^{-1}(\boldsymbol{M} - \boldsymbol{I})\boldsymbol{x}\star$, which implies $\boldsymbol{0} = \boldsymbol{I}\boldsymbol{x}\star$, or $\boldsymbol{x}\star = \boldsymbol{0}$. However, if $\boldsymbol{M} - \boldsymbol{I}$ is not invertible, there is an $n - r$ dimensional space of fixed-points, where n is the number of rows/columns in $\boldsymbol{M} - \boldsymbol{I}$ and r is the rank of that matrix. A matrix is invertible if its determinant is non-zero. For example, the determinant of the juvenile-adult model is $-fs_J$, which is not zero and so the only fixed point is at the origin.

In Python, the rank, inverse, and determinant of a matrix `B` are given by `numpy.linalg.matrix_rank(B)`, `numpy.linalg.inv(B)`, and `numpy.linalg.det(B)`.

Time-dependent solution

Four approaches:

recursion $\boldsymbol{x}(1) = \boldsymbol{M}\boldsymbol{x}(0)$, then $\boldsymbol{x}(2) = \boldsymbol{A}\boldsymbol{A}\boldsymbol{x}(0)$, and in general $\boldsymbol{x}(t) = \underbrace{\boldsymbol{M}...\boldsymbol{M}}_{t-\text{times}}\boldsymbol{x}(0)$.

matrix powers We can define matrix powers (in Python: `numpy.linalg.matrix_power`), so that $\boldsymbol{x}(t) = \boldsymbol{M}^t\boldsymbol{x}(0)$. This is efficient but doesn't provide much insight.

diagonalization Gain insight by *diagonalizing* $\boldsymbol{M} = \boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}$, where $\boldsymbol{S}$ is a matrix whose columns are the eigenvectors of $\boldsymbol{M}$ and $\boldsymbol{D}$ is a matrix with eigenvalues on the diagonal and zeros everywhere else. Substituting into the matrix power equation, $\boldsymbol{x}(t) = \left(\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}\right)^t\boldsymbol{x}(0) = \underbrace{\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}...\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}}_{t-\text{times}}\boldsymbol{x}(0) = \underbrace{\boldsymbol{S}\boldsymbol{D}^t\boldsymbol{S}^{-1}}_{t-\text{times}}\boldsymbol{x}(0)$, because the $\boldsymbol{S}^{-1}\boldsymbol{S}$ terms cancel.

series Let $\boldsymbol{c} = \boldsymbol{S}^{-1}\boldsymbol{x}(0)$. This allows us to write $\boldsymbol{x}(t) = \sum_i c_i\lambda_i^t\boldsymbol{v}_i$, where c_i, λ_i, and $\boldsymbol{v}_i$ is the ith element of $\boldsymbol{c}$, eigenvalue, and eigenvector respectively. This form also lets us see the importance of the dominant eigenvalue (i.e. eigenvalue with largest absolute value) – because all the eigenvalues get raised to the power of time, as time increases all other terms except for the dominant become neglible. Therefore, for t sufficiently large, $\boldsymbol{x}(t) \approx c_1\lambda_1^t v_1$, where λ_1 is the dominant eigenvalue. What happens when $\lambda_1 = 1$? What happens when $\lambda_1 = \lambda_2$? Try it out in Python .

change of variables Let $\boldsymbol{y}(t) = \boldsymbol{S}^{-1}\boldsymbol{x}(t)$. Then the model becomes $\boldsymbol{y}(t+1) = \boldsymbol{D}\boldsymbol{y}(t)$. But since $\boldsymbol{D}$ is diagonal, this model is exceptionally simple. It is actually just a bunch of decoupled univariate models (Why?) and you know how to handle those.

Eigen-tips (mostly for the 2 by 2 case)

If $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, determinant is $\Delta = a_{11}a_{22} - a_{12}a_{21}$, trace is $T = a_{11} + a_{22}$, and eigenvalues obey $\lambda_1 + \lambda_2 = T$ and $\lambda_1\lambda_2 = \Delta$. This leads to the characteristic polynomial $\lambda_i^2 - T\lambda_i + \Delta$. And so the eigenvalues obey $\lambda_i = (T \pm \sqrt{T^2 - 4\Delta})/2$. Finally, if $\boldsymbol{v}_i$ and λ_i are an eigenvector/eigenvalue pair for $\boldsymbol{M}$, then $\boldsymbol{M}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$ (i.e. a matrix and a single scalar value do the same thing to an eigenvector!).

Example: for the juvenile-adult model, we have $\lambda_i = (s_A \pm \sqrt{s_A^2 + 4s_J f})/2$. For each eigenvalue, solve $\begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ to find the eigenvectors. For the dominant eigenvalue, this is

$$\begin{aligned} f v_2 &= \frac{s_A + \sqrt{s_A^2 + 4s_J f}}{2} v_1 \\ s_J v_1 + s_A v_2 &= \frac{s_A + \sqrt{s_A^2 + 4s_J f}}{2} v_2 \end{aligned}$$

Could keep going but you get the idea. Simplify this system. Do the same for the other eigenvalue. Write down a time-dependent solution for this model with your computations. What are the conditions for stability of the fixed point at the origin?

Affine model

Multivariate bucket/line-up: $\boldsymbol{x}(t+1) = \boldsymbol{b} + \boldsymbol{M}\boldsymbol{x}(t)$. For fixed points solve $\boldsymbol{x}\star = \boldsymbol{b} + \boldsymbol{M}\boldsymbol{x}\star$. If $\boldsymbol{M} - \boldsymbol{I}$ is invertible, then the solution is $\boldsymbol{x}\star = (\boldsymbol{I} - \boldsymbol{M})^{-1}\boldsymbol{b}$ Note the similarity to the affine case for univariate models … . Same stability conditions as in the linear case. Can you reparameterize this model such that the fixed point is a parameter? It would be nice to just read off the fixed point woudn't it?