Multivariate nonlinear discrete-time deterministic models

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Basic model: $\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t))$, where $\boldsymbol{f}$ is a vector-valued function of a vector-valued state variable, $\boldsymbol{x}$, and t takes on discrete values, $0, 1, \ldots$. This is fairly abstract notation so let's write it out using scalar notation for a particular example. We use the susceptible-infected (S-I) model,

$$
\begin{aligned}
S(t+1) &= S(t) + m(N - S(t)) - S(t)\left(1 - e^{-\beta I(t)}\right) \\
I(t+1) &= I(t) + S(t)\left(1 - e^{-\beta I(t)}\right) - (m+\gamma)I(t)
\end{aligned}
$$

Parameters: m, birth/death rate ($1/m$ is the average lifespan measured in units of Δt); γ, recovery rate ($1/\gamma$ is the average length of infectivity, ditto); β is the contact rate (new infections per susceptible per infective per time period); N is the population size (can be rescaled to proportion of population if we rescale β as well, unless we are dealing with a discrete-population model). We return to this example throughout these notes.

Fixed points

$\boldsymbol{x}\star$ is a fixed point if $\boldsymbol{x}\star = \boldsymbol{f}(\boldsymbol{x}\star)$. Unlike the linear case, we can say very little about what these solutions will look like in general, so we look at the epidemiological special case. At equilibrium, the susceptible equation becomes $0 = m(N - S) - S(1 - e^{-\beta I})$, while the infected equation is $0 = S(1 - e^{-\beta I}) - (m+\gamma)I$. In this application, a disease-free equilibrium, in which $I = 0$, is of particular interest. In this case, the susceptible equation reduces to $S = N$.

Stability

The stability of fixed points is investigated using the *Jacobian matrix*, which for the general 2×2 case is,

$$
\begin{pmatrix}
\frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\
\frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2}
\end{pmatrix}
$$

A fixed point is stable if all of the eigenvalues of this matrix (evaluating at the fixed point) have absolute value less than one.

The Jacobian for the epidemiology model is,

$$
\begin{pmatrix}
\frac{\partial S(t+1)}{\partial S(t)} & \frac{\partial S(t+1)}{\partial I(t)} \\
\frac{\partial I(t+1)}{\partial S(t)} & \frac{\partial I(t+1)}{\partial I(t)}
\end{pmatrix}
$$

which in general is equal to,

$$
\begin{pmatrix}
-m + e^{-\beta I} & -\beta S e^{-\beta I} \\
(1 - e^{-\beta I}) & 1 + \beta S e^{-\beta I} - (m+\gamma)
\end{pmatrix}
$$

and at the disease-free equilibrium the Jacobian is,

$$
\begin{pmatrix}
1 - m & -\beta N \\
0 & 1 + \beta N - (m+\gamma)
\end{pmatrix}
$$

Since this is an upper triangular matrix, its eigenvalues are on the diagonal. Thus the stability conditions for the model are,

- $|1 - m| < 1$
- $|1 + \beta N - (m+\gamma)| < 1$

The first condition puts a hard requirement on m for stability, which is that m must be between zero and two. The other condition is more complex.