Univariate linear discrete-time deterministic models

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Basic model: $N(t+1) = f(N(t))$, where f is a linear function and t is an integer. Typically, state variable (N) is continuous. What are the units of time? When does discrete time make sense?

We will use this extremely simple model to illustrate our general approach to dynamical modelling, which is,

1. solve for equilibria/find long-term solutions
2. evaluate stability of equilibria
3. possibly evaluate for small N (near system boundaries)
4. solve for time-dependent solution or simulate various special cases through time

Geometric growth (decay)

Simplest possible discrete-time deterministic (**DD) model. Homogeneous : $f(N) = RN$ (sometimes stated as $f(N) = (1+r)N$). Solve recursion analytically (can you do this?). Now we know everything about the dynamics. Suppose $N(0) > 0$, $R > 0$. What happens if $N(0) < 0$? model of debt? What happens if $R < 0$?

(Even this ridiculously simple rule — or generalizations of it — is the basis of serious modeling in conservation biology.)

If $|R| < 1$, $N \to 0$ but $N = 0$ only in the limit, unless it starts there. A value of N such that $f(N^*) = N^*$ is called an equilibrium (or a fixed point).

Stability : what happens for perturbations in the neighborhood of the fixed point? Consider displacing the population away from N^* by δ, where $\delta \ll 1$; what happens? If $|R| < 1$ then then N will return to $N*$.

$$f(N^* + \delta) = f(N^*) + \delta f'(N)|_{N=N^*} + \delta^2/2 f''(N)|_{N=N^*} + \dots$$

Therefore the deviation $\delta \to \delta f'(N^*)$. **If $|g| \equiv |f'(N^*)| < 1$ then the deviation from the equilibrium decreases geometrically with time**:

$N = 0$ is always an equilibrium, stable (attractor , limit set) iff $|R| < 1$.

Affine models

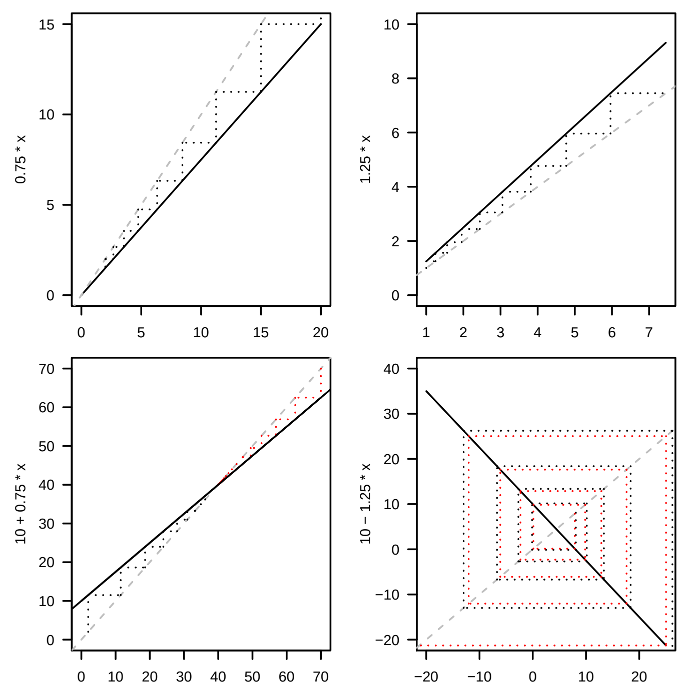
Now suppose (as in the example in the book) we are adding or subtracting a fixed amount per time step: $N(t+1) = a + bN(t)$.

Finding the equilibrium and assessing stability is straightforward.

As before we can work out the recursion. Summing the series for t steps gives $a(1-b^t)/(1-b) + b^t N(0)$; the limit is $a/(1-b) + \lim_{t\to\infty} b^t N(0)$. What happens if $|b| > 1$? If $|b| < 1$ we get a stable equilibrium at $a/(1-b)$. (For $0 < b < 1$ ("bucket model"): a is the supply rate, $1/(1-b)$ is the average residence time .) This equilibrium is an instance of Little's Law $L = \lambda W$ (which is much more general than this example).

Useful component for larger models. (Autoregressive model in time series analysis; sometimes used as the bottom level in food chain modeling; queuing theory.)

Graphical approaches: cobwebbing



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What if $N(t+1)$ depends on previous time steps $N(t-1)$ etc. as well as $N(t)$? Homogeneous linear equations: $\sum_{i=0}^{m} a_i N(t-i) = 0$. Plug in $N(t) = C\lambda^t$. Solve characteristic equation ...get a linear combination of geometric growth/decay, $\sum C_i \lambda_i^n$: largest *eigenvalue* dominates long-term behavior.