

Univariate nonlinear discrete-time models

Ben Bolker and Steve Walker

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Logistic model

Impose bounds on an otherwise ridiculous growth process. Begin with the geometric difference equation, $N(t + 1) - N(t) = RN(t)$. Set R equal to a decreasing linear function of $N(t)$ with x -intercept, N_{\max} , and y -intercept R_{\max} . This yields the logistic difference equation,

$$N(t + 1) - N(t) = R_{\max}N(1 - N(t)/N_{\max});$$

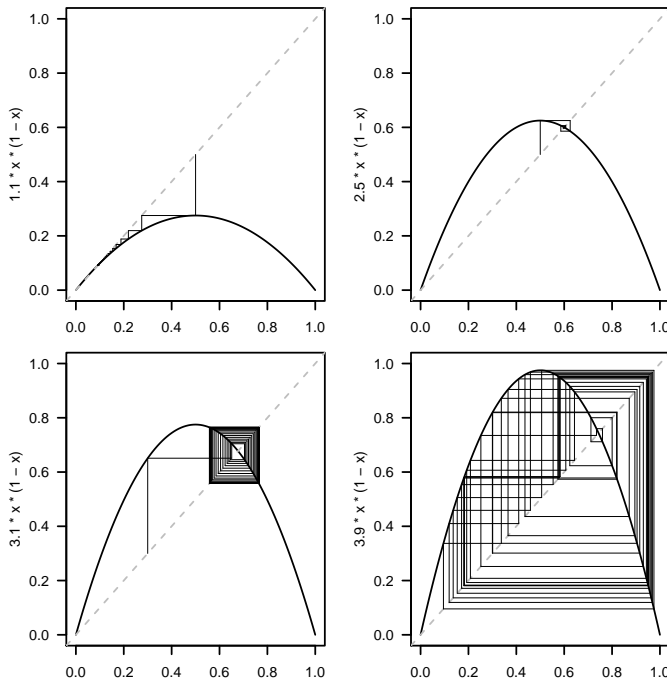
Fixed points: $N(t + 1) - N(t) = 0 = RN^*(1 - N^*/N_{\max})$ has two solutions, $N^* = 0$ and $N^* = N_{\max}$. Mathematically, we can set $N_{\max} = 1$ **without loss of generality (non-dimensionalization)**.

Stability

The geometric recursion, $N(t + 1) = f(N(t)) = RN(t)$, is stable at the fixed point $N^* = 0$, whenever $|R| < 1$. For general scalar function, f , and fixed point N^* , this criterion becomes $|f'(N)|_{N=N^*} < 1$, where $f'(N)$ is the first derivative of f with respect to N . Note that this is a true generalization because $f'(N) = R$ for the geometric model.

The derivative of the function defining the logistic recursion, $f(N) = N + R_{\max}N(1 - N/N_{\max})$, is $f'(N) = 1 + R_{\max} - 2NR_{\max}/N_{\max}$. **When are the equilibria stable?**

Bifurcation diagrams



[1, 2]

Alternative parameterizations

An ecologist or other normal person might choose to parameterize the discrete logistic model as above. A mathematician would choose $x(t + 1) = Rx(1 - x)$. The mathematician has chosen $R = r/K \rightarrow K = 1 - 1/R$. Mathematically equivalent parameterizations often have quite different meanings (or statistical properties), as well as cultural connotations. Get used to it.

More nonlinear models

Other 1-D discrete nonlinear models: *Ricker* model ($N = rNe^{-bN}$); population genetics; approximations of continuous models. Epidemic models (SI) (equivalent to discrete logistic). Metapopulation (Levins) models. (notes)

$$\begin{aligned} S(t + 1) &= m(N - S) - bSI + gI \\ &= m(N - S) - bS(N - S) + g(N - S) \\ &= m(1 - S) - bS(1 - S) + g(1 - S) \\ &= mI - bI(1 - I) + gI \\ &= (m + g - b)I + bI^2 \end{aligned} \quad (1)$$

$$\begin{aligned} N(t + 1) &= N + rN(1 - N/K) \\ &= (1 + r)N - (r/K)N^2 \\ &= (1 + r)N - rN^2 \end{aligned} \quad (2)$$

Graphical approaches, continued: *Allee effects*. Bistability, multiple stable states.

References

- [1] R. M. May (June 1976) Simple mathematical models with very complicated dynamics. *Nature*, **261**(5560):459-467.
- [2] R. M. May and G. F. Oster (1976) Bifurcations and dynamic complexity in simple ecological models. *The American Naturalist*, **110**(974):573-599.