MATH 3MB3 final sample questions

Special Instructions:

- Casio FX-991 MS or MS Plus calculator allowed, no other external aids

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- Stability condition for 2D discrete-time systems: stable if $|T| < 1 + \Delta < 2$
- Stability condition for 2D continuous-time systems: stable if $T < 0$ and $\Delta > 0$
- Jensen's inequality: $\overline{f(x)} \approx f(\bar{x}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}$
- delta method: $\sigma^2(f(x)) \approx \sigma^2(x)\left(\frac{\partial f}{\partial x}\right)^2$

Python coding

1. Write Python code that simulates a stochastic version of the Nicholson-Bailey model:

- number of hosts surviving is a binomial deviate with probability based on the escape parameter q and the number of parasitoids at the previous time step P_t:

$$S_t \sim \text{Binomial}(1 - \exp(-qP_t), V_t)$$

- number of hosts at the next time step is a Poisson deviate with mean equal to rS_t:

$$V_{t+1} \sim \text{Poisson}(rS_t)$$

- number of parasitoids at the next time step is a Poisson deviate with mean equal to $c(V_t - S_t)$:

$$P_{t+1} \sim \text{Poisson}(c(V_t - S_t))$$

Using `numpy.random.binomial(n,p)` to draw binomial deviates with number of trials `n` and probability `p` and `numpy.random.poisson(lam)` to draw Poisson deviates, write code to simulate this system for 20 time steps (including the first time step) and stores the values in a 20 × 2 numpy array. You can assume the following code has already been run:

```
import numpy as np
import numpy.random as npr
r, c, q = 2, 1.5, 0.5
V0, P0 = 5, 1
```

```
## setup code copied for convenience
import numpy as np
import numpy.random as npr
r, c, q = 2, 1.5, 0.5
V0, P0 = 5, 1
npr.seed(101) ## don't forget!
res = np.zeros((20,2))  ## set up output array
res[0,:] = [V0,P0]
for i in range(1,20): ## range goes from 1 to 19 (last row)
   ## take values from previous row
   V_prev = res[i-1,0]
   P_prev = res[i-1,1]
   ## number eaten
   S = npr.binomial(V_prev,1-np.exp(-q*P_prev))
   V = npr.poisson(r*S)
   P = npr.poisson(c*(V_prev-S))
   res[i,:] = [V,P]
```

2. A predator-prey functional response model that allows for depletion is as follows:

$$\frac{dN}{dt} = -P \cdot \frac{aN}{1 + ahN}$$

Write Python code to integrate this system of differential equations numerically, using `scipy.integrate.odeint`, for times 0, 0.1, 0.2, … 20. You can assume the following setup code has already been run.

```
import numpy as np
import scipy.integrate
params = P, a, h = 2, 1, 0.1
N0 = 10  ## initial condition
```

Remember that `scipy.integrate.odeint()` takes the arguments `func` (gradient function), `y0` (initial values), `t` (sequence of time points), in that order, and that `func` must have arguments `y` (state variables) and `t0` (time), in that order.

```
## repeat setup code
import numpy as np
import scipy.integrate
params = P, a, h = 2, 1, 0.1
N0 = 10  ## initial condition
## gradient function
def gradfun(y,t0):
   g = -P*a*y/(1+a*h*y)
   return(g)
## you could also name the argument N instead of y
tvec = np.linspace(0,20,0.1) ## set up output time vector
scipy.integrate.odeint(gradfun, N0, tvec)
```

Discrete-time models

3. Consider the deterministic version of the Nicholson-Bailey model given in question 1:

$$\begin{aligned} V_{t+1} &= rV_t e^{-qP_t} \\ P_{t+1} &= cV_t \left(1 - e^{-qP_t}\right) \end{aligned}$$

Find its equilibria and evaluate their stability. (Assume r, q, c, P_0, $V_0 > 0$.)

Solution:

V_{t+1} equation: either $V^* = 0$ **or**

$$\begin{aligned} V^* &= rV^* \exp(-qP^*) \\ 1/r &= \exp(-qP^*) \\ -\log(r) &= -qP^* \\ P^* &= \log(r)/q \end{aligned}$$

P_{t+1} equation: if $V^* = 0$ then P^* is also zero. Otherwise

$$\begin{aligned} P^* &= cV^*(1 - \exp(-qP^*)) \\ \log(r)/q &= cV^*(1 - 1/r) \\ V^* = \frac{\log(r)}{cq(1-1/r)} &= \frac{r\log(r)}{cq(r-1)} \end{aligned}$$

Stability: first find Jacobian.

$$\begin{pmatrix} r\exp(-qP) & -qrV\exp(-qP) \\ c(1-\exp(-qP)) & qcV(1-\exp(-qP)) \end{pmatrix}$$

When $V^* = P^* = 0$ this is

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$$

with eigenvalues $\{0, r\}$ so it's unstable if $r > 1$, stable if $r < 1$.

At the non-trivial equilibrium $\exp(-qP^*)$ is $1/r$, so the Jacobian is

$$\begin{pmatrix} 1 & -qV^* \\ c(1-1/r) & qcV^*(1-1/r) \end{pmatrix} = \begin{pmatrix} 1 & -r\log(r)/(c(1-1/r)) \\ c(1-1/r) & \log(r) \end{pmatrix}$$

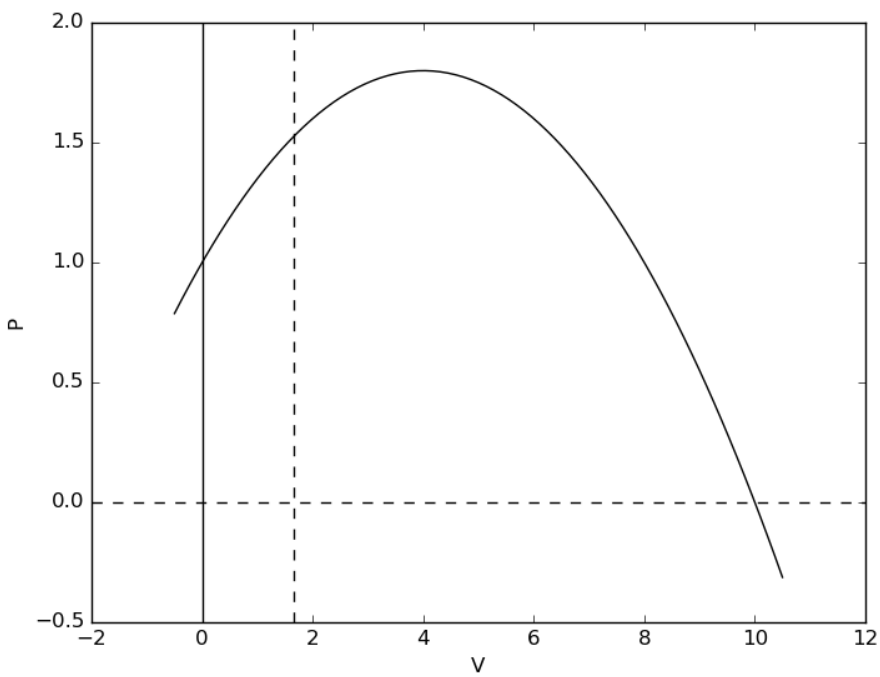
$T = 1 + \log(r)$; $\Delta = \log(r) + 1$.

The trace condition ($|T| < 1 + \Delta$) is true, but $\Delta < 1$ only if $r < 1$, so the equilibrium is stable if $r > 1$.

Continuous-time models

4. The MacArthur-Rosenzweig predator-prey model:

$$\frac{dV}{dt} = rV\left(1 - \frac{V}{K}\right) - \frac{aPV}{1 + ahV}$$
$$\frac{dP}{dt} = -mP + \frac{acPV}{1 + ahV}$$



a. Redraw the phase plane with the nullclines on your answer sheet and (1) label/identify the equilibria (*not* stability); (2) label/identify the nullclines (i.e., whether dashed nullclines and solid nullclines represent $dV/dt = 0$ or $dP/dt = 0$; (3) the qualitative direction of the vector field (i.e. whether the flow is moving left vs. right or up vs. down) in each of the four regions in the positive quadrant.

b. If V and P both have units of density, what are the units of all of the parameters?

c. Derive non-dimensional versions of the equations by applying the substitutions $\tau = mt$, $x = V/K$, $y = P/K$.

Solution:

Parameters: K= density, $r = m$=1/time, a=1/(density $\times$ time), h = time.

The solid line is the V-nullcline, the dashed line is the P-nullcline.

Non-dimensionalization: we have $t = \tau/m$, $V = xK$, $P = yK$. Substituting gives:

$$\begin{aligned}
\frac{d(xK)}{d(\tau/m)} &= r(xK)\left(1 - \frac{(xK)}{K}\right) - \frac{a(yK)(xK)}{1 + ah(xK)} \\
mK\frac{dx}{d\tau} &= rxK(1-x) - \frac{axyK^2)}{1 + ahKx} \\
\frac{dx}{d\tau} &= (r/m)x(1-x) - \frac{(aK/m)xy)}{1 + ahKx}
\end{aligned}$$

$$\begin{aligned}
\frac{d(yK)}{d(\tau/m)} &= -m(yK) + \frac{ac(yK)(xK)}{1 + ah(xK)} \\
mK\frac{dy}{d\tau} &= -mKy + \frac{acxyK^2}{1 + aKhx} \\
\frac{dy}{d\tau} &= -y + \frac{(aK/m)cxy}{1 + (aK)hx}
\end{aligned}$$

5. For the Rosenzweig-MacArthur model, compute the Jacobian. Analytically solve for all of the equilibria where V^*, P^*, or both are zero, and evaluate the stability of these equilibria (assuming all parameters are > 0).

Jacobian:

$$\begin{pmatrix} r\left(1-\frac{2V}{K}\right)-\frac{aP}{(1+ahV)^2} & \frac{-aV}{1+ahV} \\ \frac{-caP}{(1+ahV)^2} & -m+\frac{caV}{1+ahV} \end{pmatrix}$$

The simple equilibria are $\{V^*=0, P^*=0\}$ and $\{V^*=K, P^*=0\}$. At the first equilibrium we have

$$\begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}$$

with eigenvalues r, $-m$. Since all parameters are > 0, this means we have $r > 0$ so the equilibrium is unstable.

At the second equilibrium we have

$$\begin{pmatrix} -r & \frac{-aK}{1+ahK} \\ 0 & -m+\frac{caK}{1+ahK} \end{pmatrix}$$

So the eigenvalues are $-r$ (negative) and $-m+\frac{caK}{1+ahK}$; the equilibrium is unstable if $m < \frac{caK}{1+ahK}$.

Stochastic models

6. Suppose a random variable X has mean m and variance V and that $f(x) = 1-\exp(-cx)$. What is $f(\bar{X})$? Is $\overline{f(X)}$ greater than or less than $f(\bar{X})$? Why? What is the approximate value of $\overline{f(X)}$, using Jensen's inequality? What is the approximate value of the variance of $f(X)$, using the Delta method?

Solution:

$$f(\bar{X}) = 1-\exp(-cm)$$

The second derivative of f is $-c^2\exp(-cx)$ Because the second derivative of f is everywhere negative (assuming $c > 0$), $\overline{f(X)} < f(\bar{X})$ by Jensen's inequality; the upper end of the distribution is compressed by the nonlinear transformation, so large values contribute less than expected (by linearity) to the mean. Using the JI approximation,

$$\begin{aligned} \overline{f(X)} &\approx f(\bar{X}) + \frac{1}{2}\frac{\partial^2 f}{x^2}\mathrm{Var}(x) \\ &= 1-\exp(-cm) - V\cdot c^2\exp(-cm)/2 \\ &= 1-\exp(-cm)(1+Vc^2/2) \end{aligned}$$

The variance is $\left(\frac{\partial f}{\partial x}\right)^2 V = c^2\exp(-2cm)V$.

The End