Math 4MB, winter 2021, assignment 1

This assignment is due in the dropbox on Avenue to Learn by midnight (11:59 PM) on **Friday January 29**. You need to submit two files: (1) a **source file** (Jupyter notebook [`.ipynb`] or Rmarkdown [`.Rmd`] or Sweave [`.Rnw`]) and (2) an **output file** (PDF).

In this assignment you will analyze the SIRS equation, which represents an infectious disease with temporary or *waning* immunity.

Feel free to use web resources, but (1) report on any web resources you use significantly (you don't have to list resources that you look at but don't turn out to be useful); (2) maybe **don't** look at a complete solution to the problem below (i.e. a fully worked analysis of the SIRS equations).

You can use a computer algebra system (CAS: Mathematica/Maple/sympy/etc.) if you like (again, let me know what you've used), but I usually find it easiest to try the calculations by hand first and then check them with the CAS. Similarly, I strongly recommend that you write all of your equations/derivations out by hand, **then** translate them into LaTeX.

1. Suppose that infection occurs by the usual bilinear process (i.e., incidence $\propto$ the product of S and I; use β to denote the *contact rate*, i.e. the proportionality constant); that the average length of the infectious period is $1/\gamma$; and that the average duration of immunity is $1/\phi$. Write the gradients in LaTeX.

2. (Python/R) Write a function that takes arguments `t` (time), `y` (state), and `parms` (parameters) and returns a list representing the gradient vector (in Python) or a **list containing a numeric vector** (in R). (The gradients are independent of time in this case (we are modeling an *autonomous* system), but the built-in ODE solvers in Python and R expect this argument for generality.)

   (References: scipy.integrate.solve_ivp docs, deSolve::ode docs)

   Use your code to check that, for parameters $\{\beta = 2, \gamma = 1, \phi = 0.1\}$ and state vector $\{S = 0.7, I = 0.05, R = 0.25\}$, you get the correct gradient vector $\{S' = -0.045, I' = 0.02, R' = 0.025\}$. If not, **go back and check your work for the first two questions** and ask for help if necessary.

3. (Analysis) Assuming that the population size is fixed to 1 ($S + I + R = 1$), rewrite the equations as a two-dimensional system.

4. Using the two-dimensional version of the system, **argue** (you don't need to prove!) that provided $S(0) \geq 0$, $I(0) \geq 0$, $S(0) + I(0) \leq 1$, the solutions of the system remain non-negative.

5. Find the equilibria of the 2D system analytically. Confirm that the dimensions of your answers are correct.

6. Write a Python/R function that takes `parms` as an argument and returns a **list** of equilibria (expressed as vectors/arrays).

Check that for the parameter set given above, one of the equilibria is $\{S^* = 0.5, I^* \approx 0.045\}$. If not, **go back and check your answers**.

7. (Analysis) Write down the Jacobian of the 2-dimensional system. (This matrix

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right)$$

would be expressed in LaTeX like this:

```
\left(
 \begin{array}{cc}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
  \end{array}
\right)
```

8. (R/Python) Write a function that takes parameters `t`, `params`, `state` and returns the Jacobian as a 2×2 matrix

Check that the results for the parameters and state given above are as follows:

$$\left(\begin{array}{cc} -0.2 & -1.5 \\ 0.1 & 0.4 \end{array} \right)$$

if not, go back and check your work for Q6 and Q7

9. (Analysis) Show that the disease-free equilibrium (DFE) is locally asymptotically stable if $\beta < \gamma$. Interpret this result in biological terms.

10. (R/Python) Using your Jacobian function, compute the eigenvalues of the Jacobian at the DFE for three different sets of parameter values where you expect the DFE to be (1) stable (2) neutral (3) unstable.

11. (Analysis) plug the endemic equilibrium (EE: i.e. the equilibrium for which $I^* > 0$) into the (analytical) expression for the Jacobian (**for best results**, use the value of S^* with the parameters substituted, but leave I^* as a symbol). Determine a criterion for I^* to be

positive. Using this criterion, use the **trace-determinant criterion** (see Figure 1) to establish the conditions under which the endemic equilibrium is locally asymptotically stable. (Don't worry about spiral vs. non-spiral!)

12. (R/Python) Numerically evaluate the Jacobian at the EE for one set of values where you expect the EE to be stable; explain whether the results match your expectation.

13. (R/Python) Numerically integrate the ODEs for some reasonable starting values, for two sets of parameters (one where you expect the DFE to be stable and one where you expect it to be unstable). Draw plots showing the time evolution of the values and the phase plane for each case.

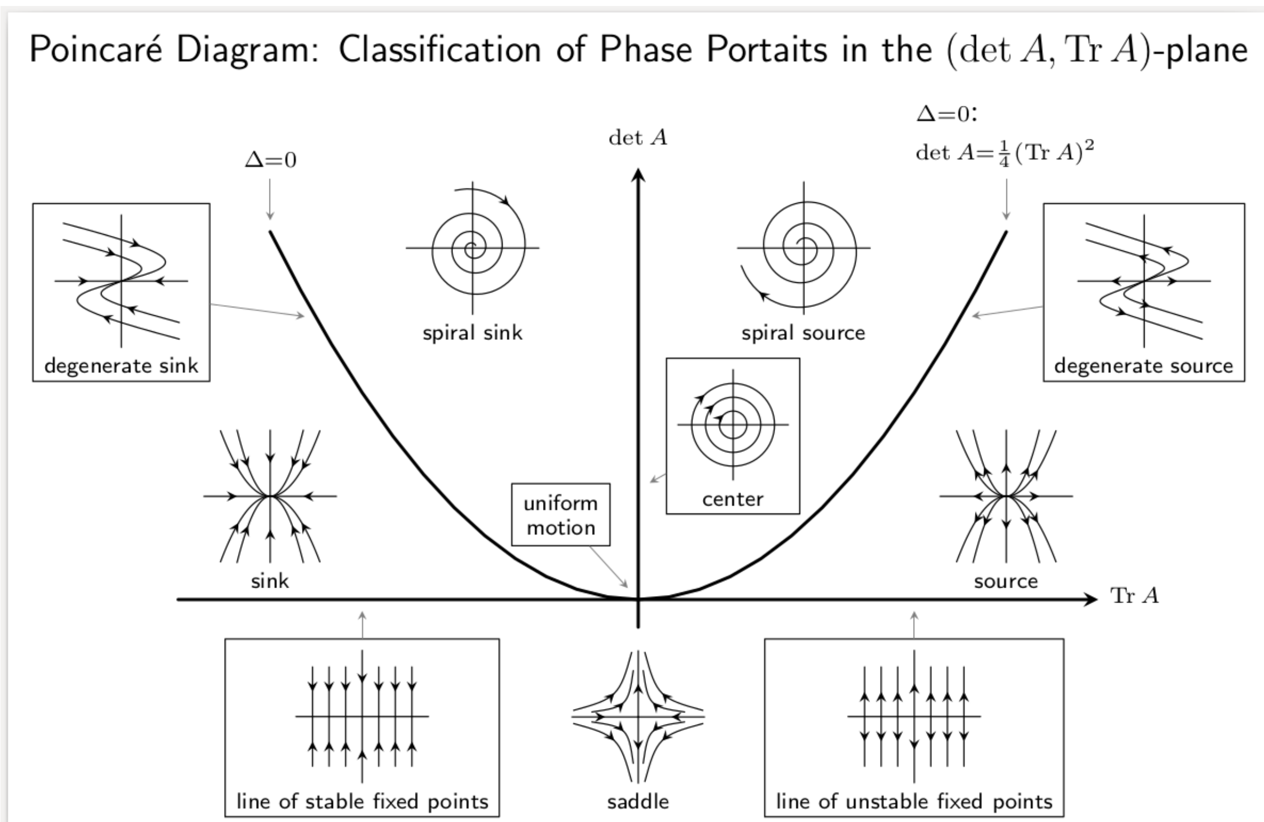

Figure 1: Diagram by Gernot Salzer on LaTeX Stack Exchange